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AGGREGATION THEORY AND ITS APPLICATION TO MODELING
IN MATHEMATICAL PROGRAMMING

by

ARTHUR M. GEOFFRION

December 1977

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Abstract

This paper reviews the newly emerging theory of a priori error bounds for aggregation of mathematical programming models. The main technical building blocks are described in some detail, and the available applications are briefly reviewed.



A. AGGREGATION THEORY AND ITS APPLICATION TO MODELING
IN MATHEMATICAL PROGRAMMING

Aggregation is very commonly used when designing mathematical programming models. One is tempted to say that nearly all models are aggregated in one way or another, as by combining individual manufactured items into product groups, employees into manpower classes, machines into machine centers, customers into customer zones, time into time periods, and so on.

There are at least two major reasons for the ubiquitous use of aggregation: it simplifies the data requirements of model-building, and it simplifies the task of solving the model once it is built. Data development is simplified because fewer data elements are needed and because "aggregate" data are more amenable to development using statistical sampling methods and summary information already available within the organization. Aggregated models are easier to solve because they are smaller, which reduces memory requirements and computing time, and they may also have a more tractable mathematical structure owing to the disappearance of complicating detail.

It is surprising that so important and commonly used a modeling device as aggregation is almost always employed in an ad hoc fashion without any theoretical justification. Educated intuition and common sense are only rarely assisted by analytical argument. It would almost seem that the old saying "modeling is an art and not a science" has frightened off any serious study of aggregation theory in modeling. Or perhaps the technical difficulty of aggregation theory in the neighboring fields of economics and statistics [10], and the apparent inapplicability of that theory to modeling in mathematical programming, has been discouraging. In any case, an effort is long overdue to help fill the void.

We wish to stress that our concern is with the modeling rather than algorithmic uses of aggregation in mathematical programming. The latter topic has been treated in depth by Lee [11] and Zipkin [12].

What is required is a way to make quantitative error estimates for alternative aggregated model designs before the model is actually built and run numerically. These error estimates would supplement common sense and should enable the design of models which constitute a provably good tradeoff between accuracy and simplicity.

More specifically, we need the following type of result. Let (P) be a fully detailed model (before any aggregation):

$$(P) \quad \begin{array}{l} \text{Maximize } f(x,y) \text{ subject to } (x,y) \in F \\ x,y \end{array}$$

where x is a vector of variables not to be aggregated and y is a vector of variables to be aggregated into a new vector z . Let (Q) be an aggregated version of (P):

$$(Q) \quad \begin{array}{l} \text{Maximize } \tilde{f}(x,z) \text{ subject to } (x,z) \in \tilde{F} \\ x,z \end{array}$$

where the objective function \tilde{f} and the feasible region \tilde{F} have been altered to accommodate the aggregation. We seek a priori bounds ϵ_1 and ϵ_2 such that

$$(1) \quad -\epsilon_1 \leq v(P) - v(Q) \leq \epsilon_2,$$

where $v(\cdot)$ denotes the optimal value of an optimization problem. By "a priori", we mean that ϵ_1 and ϵ_2 can be calculated without having to optimize (Q). We also seek a solution recovery mapping $Y(\cdot)$ from \tilde{F} into the space of y such that if (\tilde{x}, \tilde{z}) is δ -optimal^{1/} in (Q) and $\tilde{y} \in Y(\tilde{x}, \tilde{z})$, then (\tilde{x}, \tilde{y}) is $(\delta + \epsilon_1 + \epsilon_2)$ -optimal in (P). One might think of $Y(\cdot)$ as a "disaggregation rule".

All of the aggregation theoretic results to be described subsequently are of this general type.

^{1/}That is, $\tilde{f}(\tilde{x}, \tilde{z}) \geq v(Q) - \delta$. Clearly 0-optimality corresponds to the usual notion of (true) optimality.

A little reflection upon (1) and the role of the solution recovery mapping should convince the reader that they are exactly what is required for making practical decisions about which aggregated version of (P) to adopt, and then how to relate aggregate solutions back to (P) once the aggregate model is built and solved^{2/}. Notice that $\epsilon_1 + \epsilon_2$ is the most pertinent single measure of aggregation accuracy -- for instance, it is the width of the interval based on $v(Q)$ within which $v(P)$ must lie. Notice also that nothing whatever is said about how closely a solution to (Q) approximates a solution to (P) in "decision space"; all assertions are in "payoff space". We believe decision space proximity to be much less relevant an issue in a practical sense than payoff space proximity.

The model design process typically involves not one but several or even many possible choices for (Q). Let $\epsilon_1(Q)$ and $\epsilon_2(Q)$ denote the à priori error bounds as a function of (Q). Ideally, we would like the following to be true for these functions:

- a) available as explicit closed form expressions without the need for iterative numerical calculation,
- b) equal zero in the special cases where the true error is known to vanish,
- c) continuous functions of the data of (P).

Properties b) and c) together imply that the bounds are "close" to zero if (P) is "close" to any special case where the true error is known to be zero. We would also like the solution recovery mapping to be as tractable as possible.

These ideals are actually achieved for most of the applications mentioned later.

^{2/} We shall not address here the issue of how to improve the à priori bounds using information available after (Q) has been optimized, although usually this is easy to do.

Our efforts to develop this type of aggregation theory for mathematical programming have, up to the present time, revolved largely about specific applications. These applications will be reviewed (Sec. 6), but the main body of this paper is devoted to a review of the four analytical techniques which have enabled these applications.

1. FIRST ANALYTICAL TOOL: THE RESTRICTIVE APPROXIMATION THEOREM

Consider a general problem

$$(P^1) \quad \text{Maximize } f(x) \text{ subject to } x \in X$$

and the following restrictive approximation to it

$$(Q^1) \quad \text{Maximize } \tilde{f}(x) \text{ subject to } x \in X \cap \bar{X},$$

where X and \bar{X} are both subsets of the same decision universe.

Restrictive Approximation Theorem

Assume that X is not empty, that

$$(2) \quad \tilde{f}(x) \leq f(x) \text{ for all } x \text{ in } X \cap \bar{X}$$

holds, and that a Feasibility Recovery Rule $r(\cdot)$ exists which associates to every point x in X some point $r(x)$ in $X \cap \bar{X}$ in such a manner that

$$(3) \quad \tilde{f}(r(x)) \geq f(x) - \epsilon \text{ for all } x \text{ in } X$$

holds for some $\epsilon \geq 0$. Then

$$(4) \quad 0 \leq v(P^1) - v(Q^1) \leq \epsilon$$

and, for any $\delta \geq 0$, every δ -optimal solution of (Q^1) will necessarily be $(\delta + \epsilon)$ -optimal in (P^1) .

Proof. We have

$$v(Q^1) \geq \sup_{x \in X} \tilde{f}(r(x)) \geq v(P^1) - \epsilon ,$$

where the first inequality follows from the fact that $r(x) \in X \cap \bar{X}$ and the second inequality follows from (3). This, with the evident implication of (2) that $v(Q^1) \leq v(P^1)$, proves (4). It remains to show that any δ -optimal solution of (Q^1) , say x^0 , is $(\delta+\epsilon)$ -optimal in (P^1) , that is,

$$(5) \quad f(x^0) \geq v(P^1) - (\delta+\epsilon) .$$

This follows from

$$f(x^0) \geq \tilde{f}(x^0) \geq v(Q^1) - \delta \geq v(P^1) - \epsilon - \delta ,$$

where the first inequality follows from (2), the second from the definition of x^0 , and the third from (4). Q.E.D.

This result slightly generalizes a theorem first proven in [7].

2. SECOND ANALYTICAL TOOL: LAGRANGEAN DUALITY

Linear programming duality theory, its generalization to convex programming [3] and its formal application to integer programming [4], provide bounds of use in aggregation theory. The possibilities are illustrated with an example adapted from Zipkin [12].

Consider a general linear programming problem

$$(P^2) \quad \begin{array}{l} \text{Maximize } c^t x + d^t y \text{ subject to } Ax + By \leq b^t \\ x, y \geq 0 \end{array}$$

where x and y are column-vector variables and the data arrays c, d, A, B and b are conformable. The y vector is to be aggregated into a scalar variable z according to

$$(6) \quad y = \theta z; \text{ where } \theta \geq 0 \text{ is a fixed column-vector.}$$

Upon appending (6) to (P^2) and eliminating y , one obtains the aggregated problem

$$(Q^2) \quad \text{Maximize } c^t x + d^t \theta z \text{ subject to } Ax + B\theta z \leq b^t, \\ x, z \geq 0$$

which we shall presume is not infeasible.

Proportional Restriction Theorem for Linear Programs

Assume that some bound K is known such that

$$(7) \quad \sum_j y_j^* \leq K$$

holds for at least one optimal solution (x^*, y^*) of (P^2) . Corresponding to each variable y_j , let α_j be the optimal value of the trivial (continuous knapsack) problem

$$(8) \quad \text{Maximize } d_j - u^t B_j \text{ subject to } u^t B \geq d^t \theta, \\ u \geq 0$$

Define

$$(9) \quad \epsilon \triangleq \text{Max} \{0, K \text{Max}_j \{\alpha_j\}\}.$$

Then

$$(10) \quad 0 \leq v(P^2) - v(Q^2) \leq \epsilon$$

Moreover, if (x^0, z^0) is δ -optimal in (Q^2) for some $\delta \geq 0$, then $(x^0, y^0 = \theta z^0)$ will necessarily be $(\delta + \epsilon)$ -optimal in (P^2) .

Proof.

$$\begin{aligned} v(P^2) &= v(P^2 | \sum_j y_j \leq K) \text{ by hypothesis (7)} \\ &= \text{Infimum}_{u \geq 0} \left[\text{Supremum}_{x \geq 0, y \geq 0} c^t x + d^t y + u^t (b - Ax - By) \text{ subject to } \sum_j y_j \leq K \right] \\ &\quad \text{by duality theory [3]} \end{aligned}$$

$$\leq \sup_{x \geq 0, y \geq 0} c^t x + d^t y + \bar{u}^t (b - Ax - By) \text{ subject to } \sum_j y_j \leq K$$

where \bar{u} is an optimal dual solution of (Q^2)

$$= \bar{u}^t b + \sup_{x \geq 0} (c^t - \bar{u}^t A) x + \sup_{y \geq 0} (d^t - \bar{u}^t B) y \text{ subj. to } \sum_j y_j \leq K$$

Now $\bar{u}^t b = v(Q^2)$ by the definition of \bar{u} and LP duality theory, the supremum over x equals 0 because \bar{u} must satisfy $\bar{u}^t A \geq c^t$, and the supremum over y is

$$\max \{0, K \max_j \{d_j - \bar{u}^t B_j\}\}$$

because it amounts to a simple continuous knapsack problem.

Thus we have shown

$$v(P^2) \leq v(Q^2) + \max \{0, K \max_j \{d_j - \bar{u}^t B_j\}\}.$$

But $\alpha_j \geq d_j - \bar{u}^t B_j$ by (8) and the fact that

$$\bar{u}^t B_{\theta} \geq d_{\theta} \text{ and } \bar{u} \geq 0$$

hold by the definition of \bar{u} . Hence

$$v(P^2) \leq v(Q^2) + \max \{0, K \max_j \{\alpha_j\}\},$$

which proves the right-hand inequality of (10). The left-hand inequality of (10) holds because (Q^2) is really just a restriction of (P^2) .

Suppose now that (x^0, z^0) is δ -optimal in (Q^2) for some $\delta \geq 0$. Then

$$v(Q^2) - \delta \leq c^t x^0 + d^t z^0 = c^t x^0 + d^t y^0.$$

Adding to this the relation

$$v(P^2) \leq \epsilon + v(Q^2)$$

from (10), one obtains

$$v(P^2) - \delta \leq \varepsilon + c^t x^0 + d^t y^0.$$

This result shows that (x^0, y^0) is $(\delta + \varepsilon)$ -optimal in (P^2) , since (x^0, y^0) is clearly feasible in (P^2) . Q.E.D.

This theorem is one of a family of similar results which can be proved in a similar fashion. For example, there can be several simultaneous aggregations of the type (6) on different subsets of variables, there can be assumptions different from (7) concerning y^* , and (6) could be replaced by a more general linear relation in which θ is a nonnegative matrix and z is a vector.

3. THIRD ANALYTICAL TOOL: THE OBJECTIVE FUNCTION APPROXIMATION THEOREM

Consider the general problems

$$(P^3) \quad \begin{array}{l} \text{Maximize } f(x) \text{ subject to } x \in X \\ x \end{array}$$

and

$$(Q^3) \quad \begin{array}{l} \text{Maximize } \tilde{f}(x) \text{ subject to } x \in X, \\ x \end{array}$$

where X is an arbitrary non-empty set and f and \tilde{f} are both real-valued functions bounded above on X .^{3/} The function \tilde{f} is interpreted as an approximation to the "real" objective function f .

Objective Function Approximation Theorem [5]

Let ε_1 and ε_2 be scalars (not necessarily nonnegative) satisfying

$$(11) \quad -\varepsilon_1 \leq f(x) - \tilde{f}(x) \leq \varepsilon_2 \quad \text{for all } x \in X.$$

^{3/} It is permissible for f and \tilde{f} to assume the value $-\infty$ so long as they both do it on the same proper subset of X .

Then

$$(12) \quad -\epsilon_1 \leq v(P^3) - v(Q^3) \leq \epsilon_2$$

and, for any $\delta \geq 0$, any δ -optimal solution of (Q^3) will necessarily be $(\delta + \epsilon_1 + \epsilon_2)$ -optimal in (P^3) .

4. FOURTH ANALYTICAL TOOL: PROJECTION

Consider the general problem

$$(P^4) \quad \begin{array}{ll} \text{Maximize } f(w, y) & \text{subject to } (w, y) \in G \\ w, y & w \in W, y \in Y. \end{array}$$

The "projection" of this problem "onto the space of the w variables" is defined as:

$$(Q^4) \quad \begin{array}{ll} \text{Maximize } \phi(w) & \text{subject to } w \in W \cap F \\ w & \end{array}$$

where

$$(13) \quad \phi(w) \triangleq \sup_y f(w, y) \text{ subject to } (w, y) \in G \text{ and } y \in Y$$

$$(14) \quad F \triangleq \text{any subset of } \{w: \text{ the problem defining } \phi(w) \text{ has a feasible solution}\}.$$

We adopt the convention that $\phi(w) = -\infty$ for those values of w for which the defining problem in (13) is infeasible. Note that the condition $w \in F$ is an implicit feasibility constraint which must hold for (P^4) as well as for (Q^4) .

Projection Theorem [2]

If (w^0, y^0) is ϵ -optimal in (P^4) , where $\epsilon \geq 0$, then w^0 must be ϵ -optimal in (Q^4) . If w^0 is ϵ' -optimal in (Q^4) and y^0 is ϵ'' -optimal in the problem defining $\phi(w^0)$, where $\epsilon' \geq 0$ and $\epsilon'' \geq 0$, then (w^0, y^0) is $(\epsilon' + \epsilon'')$ -optimal in (P^4) . If problem (P^4) is feasible, both (P^4) and (Q^4) have identical supremal values.

It can also be shown [2] that projection is a convexity-preserving transformation. That is, if (P^4) has a concave objective function over a convex feasible region, then the same will be true of (Q^4) .

Projection is closely related to the notion of hierarchical decisions and control (e.g., see [9]), although previously published work along these lines has not fully exploited this fact.

One of the most valuable uses of projection is as a device to facilitate the application of each of the three previously described devices. This is illustrated in the next section.

5. OBJECTIVE FUNCTION APPROXIMATION CUM PROJECTION

The four analytical tools are building blocks which can be combined in various ways to obtain various aggregation theoretic results. In this section we show how projection and objective function approximation can be combined in a fruitful manner.

Start with a feasible problem of the form

$$(P^5) \quad \begin{array}{l} \text{Maximize } f(x)+g(y) \text{ subject to } (x,y) \in W, x \in X, y \in Y \\ x,y \end{array}$$

into which some new aggregation variables z are introduced along with an associated set of definitional constraints $z = Z(x,y)$:

$$(P^6) \quad \begin{array}{l} \text{Maximize } f(x)+g(y) \text{ subject to } (x,y) \in W, x \in X, y \in Y, z = Z(x,y). \\ x,y,z \end{array}$$

For instance, each component of z might be defined as the sum of selected components of y . Since the aim is to replace y by the aggregate variables z , project (P^6) onto the space of the x and z variables:

$$(P^7) \quad \begin{array}{l} \text{Maximize } f(x)+\phi(x,z) \text{ subject to } x \in X \text{ and } (x,z) \in F \\ x,z \end{array}$$

where

$$(15) \quad \phi(x,z) \triangleq \sup_y g(y) \text{ subject to } (x,y) \in W, y \in Y, z = Z(x,y)$$

$$(16) \quad F \triangleq \text{a suitable subset of } \{(x,z): \text{ the problem defining } \phi(x,z) \text{ has a feasible solution}\}.$$

By a "suitable" subset in (16) we mean F should be chosen, if possible, so as not to spoil whatever special structure (P^7) may have that could be of use in solving the aggregate problem. Finally, let us replace ϕ in (P^7) by some more tractable approximation $\tilde{\phi}$ to obtain

$$(Q^5) \quad \begin{array}{l} \text{Maximize } f(x) + \tilde{\phi}(x,z) \\ \text{subject to } x \in X \text{ and } (x,z) \in F. \\ x, z \end{array}$$

What is the relationship between the original problem (P^5) and the aggregated problem (Q^5) ? The answer can be found by examining the successive steps of the transformation

$$(P^5) \rightarrow (P^6) \rightarrow (P^7) \rightarrow (Q^5)$$

in light of the results established in the preceding sections. The relationship between (P^5) and (P^6) is obvious, although technically (P^5) could be viewed as a projection of (P^6) . The relation between (P^6) and (P^7) is available from the Projection Theorem (the role of w is played by (x,z)). The relation between (P^7) and (Q^5) is available from the Objective Function Approximation Theorem. Putting it all together, we obtain the following end result.

Theorem

Suppose that $f + \tilde{\phi}$ is bounded above on the set of feasible solutions of (Q^5) , and that $f + \tilde{\phi}$ equals $-\infty$ if and only if $f + \phi$ equals $-\infty$ on a (possibly

empty) proper subset of this set. Suppose further that scalars ϵ_1 and ϵ_2 exist such that

$$-\epsilon_1 \leq \phi(x, z) - \tilde{\phi}(x, z) \leq \epsilon_2$$

holds for all (x, z) such that ϕ is finite and feasible in (Q^5) . Then

$$(17) \quad -\epsilon_1 \leq v(P^5) - v(Q^5) \leq \epsilon_2.$$

Moreover, if (x^0, z^0) is δ -optimal in (Q^5) for some $\delta \geq 0$ and $y^0 \in Y$ satisfies $(x^0, y^0) \in W$ and

$$(18) \quad g(y^0) \geq \tilde{\phi}(x^0, z^0) - \epsilon_1,$$

then (x^0, y^0) is $(\delta + \epsilon_1 + \epsilon_2)$ -optimal in (P^5) .

Proof. The Objective Function Approximation Theorem applied to (P^7) and (Q^5) yields

$$(19) \quad -\epsilon_1 \leq v(P^7) - v(Q^5) \leq \epsilon_2.$$

But $v(P^5) = v(P^7)$ by the Projection Theorem, so (17) is at hand. Suppose that (x^0, z^0) is δ -optimal in (Q^5) and that $y^0 \in Y$ satisfies $(x^0, y^0) \in W$ and (18). Then (x^0, y^0) is feasible in (P^5) and

$$\begin{aligned} f(x^0) + g(y^0) &\geq f(x^0) + \tilde{\phi}(x^0, z^0) - \epsilon_1 \quad \text{by (18)} \\ &\geq v(Q^5) - \delta - \epsilon_1 \quad \text{by the } \delta\text{-optimality of } (x^0, z^0) \text{ in } (Q^5) \\ &\geq v(P^5) - \epsilon_2 - \delta - \epsilon_1 \quad \text{by (17)} \\ &= v(P^5) - (\delta + \epsilon_1 + \epsilon_2), \end{aligned}$$

which shows that (x^0, y^0) is $(\delta + \epsilon_1 + \epsilon_2)$ -optimal in (P^5) .

Q.E.D.

6. APPLICATIONS

Most applications to date have been in the area of distribution system planning and related models, but recently results have also been achieved for a production scheduling model and a production/marketing model.

In [7] we applied the Restrictive Approximation Theorem to achieve \hat{a} priori error bounds for customer aggregation in a class of distribution system design models. Customer aggregation is universally practiced in this context because most firms have many tens of thousands of customers; to incorporate all of them individually would lead to excessive data development costs, inordinate computing time and primary storage requirements, and an excessive degree of solution detail. Our error bounds deal with the most natural type of customer aggregation, namely pro rata by demand, for single commodity one or two-stage distribution systems with possible economies-of-scale in facility throughput costs. A companion paper [8] extends these results to the multicommodity case with the help of an auxiliary model based on Lagrangean relaxation. It turns out that commodity aggregation error bounds are also obtained as a byproduct of the analysis.

Zipkin [12] was able to rederive the \hat{a} priori bound of [7], for the special case of the classical transportation problem, using the approach of Sec. 2. He also presents \hat{a} priori and \hat{a} posteriori error bounds for more general network models aggregated in an imaginative variety of ways.

Another paper [6] applies Projection and the Objective Function Approximation Theorem to obtain \hat{a} priori error bounds for a class of procurement problems in a fairly general logistical setting. The items being aggregated were the various commodities being procured. An interesting

aspect of the results obtained is that linear programming can be used to determine the best aggregated model once the items to be aggregated are specified, where "best" means that values of certain arbitrary coefficients in the model are determined so as to yield the smallest *a priori* error bound. This is apparently the first instance where LP has been used to determine an optimal aggregative approximation to a class of mathematical programming problems.

In some unpublished notes we have applied Projection and the Restrictive Approximation Theorem to obtain *a priori* error bounds for product aggregation in a machine loading and scheduling model now being implemented for an injection molding company. The model involves several thousand products made using several hundred tools (mold bases) mounted on a hundred machines. The aim is to decide how much of each product to make and to sell during each time period over a full season, and also to decide how best to utilize tools and machines. There are constraints on tool and machine availability, the maximum production rate, and minimum and maximum sales. The costs for machine operation, inventory, and sales are linear but production costs are concave. *A priori* error bounds and solution recovery results enable the problem size to be reduced while preserving the special structure on which the specialized solution technique depends (this technique uses Lagrangean relaxation [4] in conjunction with an efficient network flow optimization code [1]).

In another set of unpublished notes we applied the Restrictive Approximation Theorem to regional aggregation in a comprehensive planning model now being implemented by a large agricultural supply cooperative. The model must determine how much of each commodity to produce where, including trades and exchanges with competitors, and how great a market share to obtain in each

marketing region via controllable market development expenditures. There are pervasive nonlinearities in the market development cost functions, but otherwise the model is of linear programming type (more precisely, it could be viewed as a generalized network with side constraints). Our aggregation theoretic results apply to the selective coarsening of the original grid of customer regions, which was taken for convenience as a checkerboard of square plots 50 miles on a side.

7. CONCLUSION

The results achieved to date show clearly that *à priori* error bounds for aggregation in mathematical programming models is a topic which is both tractable mathematically and applicable to problems of practical interest. Perhaps the greatest need at this time is for empirical studies aimed at assessing the quality of the *à priori* bounds. Favorable indications here would justify the development of additional theoretical tools and an effort to popularize aggregation theory as a model design technique.

On the basis of the preliminary computational evidence currently available, it would appear that *à priori* bounds often show that negligible error is associated with quite substantial aggregations. Frequently these aggregations are ones an experienced modeler would have accepted on intuitive grounds anyway, but he should nevertheless be pleased to have a solid rationale for use when defending the model's assumptions. As the extent of aggregation becomes still greater, the empirical error may for a time remain acceptably small even though the *à priori* bound becomes unacceptable. Effective model design in this range awaits the development of *à priori* error estimators aimed at probable range rather than worst case.

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